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ASYMPTOTIC MODELS OF INTERNAL STATIONARY WAVES

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Equations of stationary long waves on the interface between a homogeneous fluid and an exponentially stratified fluid are considered. An equation of the second-order approximation of the shallow water theory inheriting the dispersion properties of the full Euler equations is used as the basic model. A family of asymptotic submodels is constructed, which describe three different types of bifurcation of solitary waves at the boundary points of the continuous spectrum of the linearized problem.

Key words: *two-layer fluid, exponential stratification, solitary waves.*

Introduction. Equations of an inviscid two-layer fluid with a piecewise-constant density experiencing a jump at the interface between the layers are used (see [1, 2]) as a mathematical model of internal waves in a pynocline. In this model, solitary waves and smooth bores are described by the equation of the second-order approximation of the shallow water theory, which was derived by Ovsyannikov [3]. Such an equation was obtained in [4] for the case with no slipping of the layers in the main flow. A similar approximation for long waves in a fluid with a piecewise-constant Brunt–Väisälä frequency was considered in [5]. It was noted [6] that asymptotic series for solitary waves are highly sensitive to small perturbations of the density field. The main objective of the present work is to estimate the influence of weak continuous stratification on the parameters of nonlinear waves on the interface. The behavior of the critical parameters of wave motion with stratification vanishing in one of the layers is studied. The density of the fluid in the second layer is assumed to be constant. The basic feature of this limit transition is the concentration of the spectra of higher modes in a narrow band of the boundary-layer type in the plane of bifurcation parameters. The presence of such a layer substantially affects the asymptotic behavior of solitary waves of the leading mode. In particular, a region of parameters is found, where the branching of solutions at the points of the spectrum boundary differs from the bifurcation of solitary waves to plateau- and bore-type waves for the model of a two-layer fluid [7, 8].

1. Initial Equations. Plane stationary flows of an ideal incompressible inhomogeneous fluid are described by the Euler equations

$$\rho(uu_x + vu_y) + p_x = 0, \qquad \rho(uv_x + vv_y) + p_y = -\rho g,$$
$$u_x + v_y = 0, \qquad u\rho_x + v\rho_y = 0, \tag{1}$$

where ρ is the fluid density, u and v are the velocity-vector components, and p is the pressure. We consider a two-layer flow in the region bounded by a rigid horizontal bottom $y = -h_1$ and by a top cover $y = h_2$ (Fig. 1). The interface has the shape $y = \eta(x)$, the value $\eta = 0$ corresponding to the waveless regime. As $x \to -\infty$, the velocity vector of the fluid (u, v) in the jth layer tends to a constant vector $(u_j, 0)$ (u_j is the wave velocity with respect to the corresponding layer; $j = 1, 2$). If the stream function ψ is introduced for the velocity field $u = \psi_y$ and $v = -\psi_x$, system (1) reduces to the second-order quasi-linear elliptical equation (Dubreil-Jacotin–Long equation) [1, 2]

$$\rho(\psi)(\psi_{xx} + \psi_{yy}) + \rho'(\psi)(gy + |\nabla\psi|^2/2) = H'(\psi).$$

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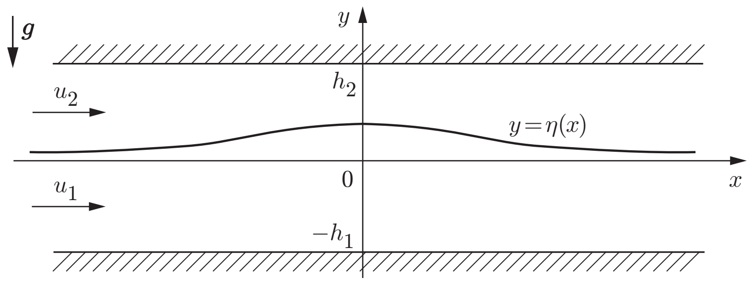

Fig. 1. Schematic of fluid motion.

Here $H(\psi) = \rho(\psi)b(\psi)$, where $b(\psi)$ is the Bernoulli function; the dependence of the fluid density on the stream function is defined by the density distribution along the streamlines in the undisturbed flow: $\rho(\psi) = \rho_\infty(\psi/u_j)$ in the layer with the number j. The expression for density in the main flow is assumed to be

$$\rho_\infty(y) = \begin{cases} \rho_1, & -h_1 < y < 0, \\ \rho_2 \exp(-N^2 y/g), & 0 < y < h_2, \end{cases}$$

where $N = \text{const}$ is the Brunt–Väisälä frequency, and the constants $\rho_j > 0$ satisfy the inequality $\rho_2 < \rho_1$. For $N = 0$, this density distribution ensures a usual two-layer stratification with constant densities in the layers; for $N \neq 0$, the Bernoulli function b is constant in the lower layer ($b = u_1^2/2$) and has the following form in the upper layer:

$$b(\psi) = \frac{1}{2}\,u_2^2 + \frac{g\psi}{u_2} + \frac{g^2}{N^2}\Big(1 - \exp\Big\{\frac{N^2\psi_2}{gu_2}\Big\}\Big).$$

The stream function is normalized to the condition $\psi = 0$ on the interface. Then, the boundary condition $\psi(x, -h_1) = -u_1 h_1$ should be satisfied on the bottom, and the condition $\psi(x, h_2) = u_2 h_2$ should be satisfied on the rigid cover. By virtue of the Bernoulli equation

$$|\nabla\psi|^2/2 + p/\rho(\psi) + gy = b(\psi),$$

the pressure p is expressed via the function ψ; therefore, the condition of pressure continuity on the interface $y = \eta(x)$ yields the nonlinear boundary condition for the stream function $[\rho(\psi)(|\nabla\psi|^2 + 2gy - 2b(\psi))] = 0$ (the square brackets indicate a jump in a quantity). We also note that the conservation of the total horizontal momentum of the two-layer fluid yields the integral relation

$$\int_{-h_1}^{h_2} \Big(p + \rho(\psi)\psi_y^2\Big)\,dy = C, \tag{2}$$

where the constant C is determined by the asymptotic curve of density, velocity, and condition of hydrostatic pressure in the limit flow at infinity.

2. Long-Wave Approximation. In the problem considered, stratification is determined by the dimensionless Boussinesq parameters

$$\sigma = N^2 h_2/g, \qquad \mu = (\rho_1 - \rho_2)/\rho_2,$$

where σ characterizes the density gradient in the inhomogeneous layer, and μ indicates the jump in density on the interface. In the present work, we assume that the ratio σ/μ is small and use the parameter σ as a modeling parameter. Following [9], we consider the long-wave approximation, where the ratio of the vertical and horizontal scales of motion is of the order of $\sqrt{\sigma}$. Using the undisturbed depth h_2 of the upper layer as the vertical scale and the fluid discharge in the jth layer as the scale for the stream function $\psi = \psi_j$ in this layer, we introduce the dimensionless variables

$$(\sqrt{\sigma}\,x, y, \eta) = h_2\,(\bar{x}, \bar{y}, \bar{\eta}), \qquad \psi_j = u_j h_j \bar{\psi}_j \qquad (j = 1, 2).$$

Then, the following equations should be satisfied in the lower layer:

$$\sigma\psi_{1xx} + \psi_{1yy} = 0 \qquad [-r < y < \eta(x)],$$
$$\psi_1(x, -r) = -1, \qquad \psi_1(x, \eta(x)) = 0 \tag{3}$$

($r = h_1/h_2$). (Hereinafter, the bar in the dimensionless quantities $\bar{x}$, $\bar{y}$, $\bar{\eta}$, and $\bar{\psi}_j$ is omitted.) Correspondingly, the equations for the upper layer take the form

$$\sigma\psi_{2xx} + \psi_{2yy} + \lambda^2(\psi_2 - y) = \sigma(\psi_{2x}^2 + \psi_{2y}^2 - 1)/2 \qquad [\eta(x) < y < 1],$$
$$\psi_2(x, \eta(x)) = 0, \qquad \psi_2(x, 1) = 1, \tag{4}$$

where $\lambda = Nh_2/u_2$. The dynamic condition on the interface implies the relation

$$rF_1^2(\sigma r^2\psi_{1x}^2 + r^2\psi_{1y}^2 - 1) + 2\eta = F_2^2(\sigma\psi_{2x}^2 + \psi_{2y}^2 - 1) \qquad [y = \eta(x)], \tag{5}$$

where F_j are the densimetric Froude numbers:

$$F_j^2 = \rho_j u_j^2/(g(\rho_1 - \rho_2)h_j) \qquad (j = 1, 2).$$

The dimensionless parameter λ has the meaning of the inverse densimetric Froude number for an exponentially stratified upper layer of the fluid, because it satisfies the relation $\lambda^2 = \sigma g h_2/u_2^2$. The parameters σ, μ, λ, and F_2 are related by the expression

$$\lambda^2 = \sigma/(\mu F_2^2), \tag{6}$$

which plays an important role in the analysis of the asymptotic behavior of the solution as $\sigma \to 0$. Note that the boundary condition (5) is equivalent to the integral relation

$$\mu r^3 F_1^2 \int\limits_{-r}^{\eta} \Big(\psi_{1y}^2 - \sigma\psi_{1x}^2\Big)\, dy - (1+\mu)\eta^2 + \mu r F_1^2(\eta - r)$$
$$+ \int\limits_{\eta}^{1} \mathrm{e}^{-\sigma\psi_2} \Big\{\mu F_2^2(1 + \psi_{2y}^2 - \sigma\psi_{2x}^2) - 2\sigma^{-1}(\mathrm{e}^{\sigma\psi_2} - 1) + 2(\psi_2 - y)\Big\}\, dy$$
$$= 2\mu F_2^2 + 2(\lambda^{-2} + \sigma^{-2})(1 - \sigma - \mathrm{e}^{-\sigma}), \tag{7}$$

which is the dimensionless version of Eq. (2) with the pressure eliminated by virtue of the Bernoulli equation. Constructing an asymptotic expansion for the solution of Eqs. (3) and (4) in the form $\psi_j = \psi_j^{(0)} + \sigma\psi_j^{(1)} + O(\sigma^2)$ ($j = 1, 2$) leads to the following expressions for the coefficients:

$$\psi_1^{(0)} = \frac{y - \eta}{r + \eta}, \qquad \psi_2^{(0)} = y - \eta\, \frac{\sin\lambda(1 - y)}{\sin\lambda(1 - \eta)},$$

$$\psi_1^{(1)} = -\frac{1}{6}\Big(\frac{1}{r + \eta}\Big)_{xx} \Big\{(y + r)^3 - (r + \eta)^2(y + r)\Big\},$$

$$\psi_2^{(1)} = \frac{\sin\lambda(1 - y)}{2\lambda}\Big(\frac{\eta}{\sin\lambda(1 - \eta)}\Big)_{xx} \Big\{(1 - \eta)\cot\lambda(1 - \eta) - (1 - y)\cot\lambda(1 - y)\Big\}$$
$$+ \frac{1}{6}\eta^2\Big\{\frac{\sin\lambda(\eta - y) + \sin\lambda(1 - \eta) - \sin\lambda(1 - y)}{\sin^3\lambda(1 - \eta)} + \frac{\sin^2\lambda(1 - y)}{\sin^2\lambda(1 - \eta)} - \frac{\sin\lambda(1 - y)}{\sin\lambda(1 - \eta)}\Big\} + \frac{\eta(\eta - y)}{2}\,\frac{\sin\lambda(1 - y)}{\sin\lambda(1 - \eta)}.$$

Substituting these asymptotic formulas for the functions ψ_1 and ψ_2 into Eq. (7) and truncating terms with accuracy of the order of $O(\sigma^2)$, we obtain a model of the second-order approximation of the shallow water theory. As a result, we obtain an ordinary differential equation for the function η

$$\sigma\Big(\frac{d\eta}{dx}\Big)^2 = \frac{\eta^2(A_0 + A_1\eta + A_2\eta^2 + A_3\eta^3)}{B_0 + B_1\eta + B_2\eta^2 + B_3\eta^3 + B_4\eta^4}, \tag{8}$$

where the coefficients A_j and B_j are trigonometric polynomials with respect to the variable η. These coefficients are expressed via the quantities $s_n = \sin n\lambda(1-\eta)$ and $c_n = \cos n\lambda(1-\eta)$ with integer and semi-integer values of n by the following formulas:

$$A_0 = 18r\lambda s_1^2\{[2(F_1^2-1)-\sigma F_2^2]s_1^2 + \lambda F_2^2 s_2\},$$

$$A_1 = 2\lambda F_2^2\{s_1^2[\lambda(9-2\sigma r)s_2 - s_1^2(6r\lambda^2+9\sigma)] - 2s_{1/2}^2[r\sigma\lambda s_1 + 3r\sigma\lambda^2(1+2c_1)]\} - 36\lambda s_1^4,$$

$$A_2 = 4\lambda^2 F_2^2 s_{1/2}^2\{3\sigma\lambda(r-1)(1+2c_1) - 4(3\lambda s_1^2+\sigma s_2)c_{1/2}^2 - \sigma s_1\},$$

$$A_3 = 12\lambda^3\sigma F_2^2 s_{1/2}^2(1+2c_1),$$

$$B_0 = 12\lambda r^3 F_1^2 s_1^4 + 9rF_2^2(2\lambda - s_2)s_1^2,$$

$$B_1 = 9F_2^2\{r\lambda(2\lambda-s_2)s_2 - [2(r-1)\lambda + s_2]s_1^2\},$$

$$B_2 = 9\lambda F_2^2\{\lambda[r(s_1^2-3)+2]s_2 + 4s_1^4 + 2r\lambda^2 - 2(r\lambda^2+3)s_1^2\},$$

$$B_3 = -9\lambda^2 F_2^2\{(c_1^2+2)s_2 + 2\lambda(r-1)c_1^2\}, \qquad B_4 = -18\lambda^3 F_2^2 c_1^2.$$

Equation (8) is an analog of the model proposed by Ovsyannikov [3] for a two-layer fluid with constant densities in the layers. This equation is further used as the basic model for analyzing the behavior of solutions of the solitary-wave type in the limit as $\sigma \to 0$.

3. Dispersion Properties. In nonlinear media with dispersion, solitary waves usually propagate with velocities greater than the phase velocities of linear waves. Therefore, to describe the region of supercritical values of parameters in Eqs. (3)–(5), we consider the properties of the spectrum of the problem on small perturbations in a one-dimensional flow with the stream functions $\psi_1 = r^{-1}y + w_1(x,y)$ and $\psi_2 = y + w_2(x,y)$. Linearization of the equations with respect to the functions η and w_j and construction of solutions in the form of wave packets $\eta(x) = a\exp(ikx)$ and $w_j(x,y) = W_j(y)\exp(ikx)$ yield the dispersion relation

$$\Delta(k; F, \sigma, \lambda) = 0 \tag{9}$$

with the function Δ defined by the formula

$$\Delta = F_1^2\sqrt{\sigma}\,rk\coth\sqrt{\sigma}\,rk + F_2^2\Big(\sqrt{\lambda^2-\sigma k^2-\sigma^2/4}\,\cot\,\sqrt{\lambda^2-\sigma k^2-\sigma^2/4} - \sigma/2\Big) - 1.$$

As the problem of finding the normal modes for Eqs. (3)–(5) is self-adjoint, the square of the wavenumber k^2, which plays the role of the eigenvalue in this problem, is always real. Thus, the roots k of Eq. (9) can be only real or imaginary. By virtue of the even character of the dispersion function with respect to k, these roots form symmetric pairs on the coordinate axes of the complex k plane, and only the root $k = 0$ can be multiple. The dispersion function is extended to the real domain $\lambda^2 < \sigma k^2 + \sigma^2/4$ by the real expression

$$\Delta = F_1^2\sqrt{\sigma}\,rk\coth\sqrt{\sigma}\,rk + F_2^2\Big(\sqrt{\sigma k^2-\lambda^2+\sigma^2/4}\,\coth\,\sqrt{\sigma k^2-\lambda^2+\sigma^2/4} - \sigma/2\Big) - 1.$$

The spectrum of the linearized problem is formed by the points in the plane of the Froude number pairs $F = (F_1, F_2)$ for which the dispersion relation (9) has at least one pair of real roots k. This set is symmetric with respect to the coordinate axes of the plane F, because the function Δ is even with respect to each of the parameters F_j. Depending on the number of pairs of real roots of Eq. (9) corresponding to this point F, the whole spectrum is divided into a countable set of subdomains containing spectra of individual wave modes. As the point F is disturbed, the real roots appear as a result of their transition from the imaginary axis to the real axis through the value $k = 0$; therefore, the boundaries of the modal domains are defined by the branches of the curve described by the equation

$$\Delta(0; F, \sigma, \lambda(F_2)) = F_1^2 + F_2^2\Big(\sqrt{\sigma/(\mu F_2^2) - \sigma^2/4}\,\cot\,\sqrt{\sigma/(\mu F_2^2) - \sigma^2/4} - \sigma/2\Big) - 1 = 0 \tag{10}$$

[here we take into account Eq. (6) for $\lambda = \lambda(F_2)$]. The outer boundary of the spectrum is defined by the branches that have horizontal asymptotes $F_2 = \pm F_2^{(*)}$ as $|F_1| \to \infty$, where

$$F_2^{(*)} = \sqrt{\sigma/\mu}\,\Big/\sqrt{\pi^2 + \sigma^2/4}. \tag{11}$$

The part of the spectrum with points having more than one pair of real wavenumbers is located inside the band $|F_2| < F_2^{(*)}$ whose thickness is of the order of $\sqrt{\sigma}$. In the limit, as $\sigma \to 0$, this part containing the spectra of the higher wave modes disappears. At $\sigma = 0$, there remains the spectrum of linear waves in the fluid with constant densities ρ_1 and ρ_2 in the layers. This spectrum is generated by the wavenumbers of one mode and coincides with the circle $F_1^2 + F_2^2 \leqslant 1$. It can be easily noted that the boundary of the spectrum of the disturbed problem described by Eq. (10) transforms to a unit circle nonuniformly with respect to (F_1, F_2). This nonuniformity arises because of the presence of the small parameter σ at the leading power of F_2 in the dispersion relation (9) [with allowance for Eq. (6) for λ] and, correspondingly, in Eq. (10).

Note that the presence of the small parameter σ in the left side of the differential equation (8) does not prevent uniform validity (in terms of x) of the asymptotic approximation in the plane of the flow. The reason is a special structure of the right side of Eq. (8) whose coefficients turn out to be consistent with the spectral properties of the original problem. Indeed, using the expansions of the coefficients A_j and B_j in powers of η, we can write Eq. (8) in the form

$$\sigma \eta_x^2 = \eta^2[\gamma_0 + \gamma_1 \eta + O(\eta^2)]. \tag{12}$$

Here the coefficient $\gamma_0(F, \sigma, \lambda)$ has the form $\gamma_0 = \alpha_0/\beta_0$, where

$$\alpha_0 = 12\Big\{F_1^2 + F_2^2\Big(\lambda \cot\,\lambda - \frac{\sigma}{2}\Big) - 1\Big\}, \qquad \beta_0 = 4r^2 F_1^2 + 3F_2^2\,\frac{2\lambda - \sin 2\lambda}{\lambda \sin^2 \lambda},$$

and the expression for the coefficient $\gamma_1 = (\alpha_1\beta_0 - \alpha_0\beta_1)/\beta_0^2$ involves the quantities

$$\alpha_1 = -48(F_1^2 - 1)\lambda \cot\,\lambda - F_2^2(36\lambda^2 \cot^2\lambda - 12r^{-1}\lambda \cot\,\lambda - 8\lambda^2) - 12r^{-1}$$

$$- \sigma F_2^2\Big[6r^{-1} + \Big(\frac{8\lambda \sin^2(\lambda/2)}{\sin^3\lambda} - \frac{64}{3}\Big)\lambda \cot\,\lambda + \frac{4\lambda \sin^2(\lambda/2)}{3\sin^3\lambda}\Big(1 + \frac{3\lambda}{\sin\lambda}\Big)\Big],$$

$$\beta_1 = -16r^2 F_1^2 \lambda \cot\,\lambda - 6F_2^2\Big(2 - \frac{2\lambda - \sin 2\lambda}{2r\lambda \sin^2\lambda}\Big).$$

It should be noted that it is the dependence of the coefficient γ_0 on the parameters F_j that contains information on the dispersion properties of the original problem. Indeed, as the value of β_0 is rigorously positive for $F_1^2 + F_2^2 \neq 0$, and the relation $\alpha_0 = 12\Delta + O(\sigma^2)$ is valid for α_0, the pattern of the zero-level lines $\gamma_0(F, \sigma, \lambda(F_2)) = 0$ reproduces both the shape and the multimodal structure of the spectrum obtained for the basic Euler equations with accuracy to $O(\sigma^2)$. The solitary waves of the leading mode branching off from the waveless regime at the boundary points of the spectrum are necessarily supercritical in the sense of satisfaction of the inequality $\gamma_0 > 0$. Note, for points F located outside the spectrum and satisfying the inequality $\gamma_0(F, \sigma, \lambda(F_2)) > 0$, the parameter λ takes the values in the interval $0 < \lambda < \pi$. In the supercritical domain external with respect to the spectrum, all roots of the dispersion relation (9) are imaginary. Let $k = \pm i\kappa$ $(\kappa > 0)$ be the pair of roots closest to the point $k = 0$. The dimensionless parameter $\varepsilon = \sqrt{\sigma}\,\kappa$ yields the indicator of exponential decay of solitary waves in the initial dimensional variables: $\eta(x) = O(\exp(-\varepsilon|x|/h_2))$ as $|x| \to \infty$. It follows from the expansion of the dispersion function

$$\Delta(0; F, \sigma, \lambda) - \Delta(i\kappa; F, \sigma, \lambda) = \Big(\frac{1}{3}\, r^2 F_1^2 + F_2^2\,\frac{2\lambda - \sin 2\lambda}{4\sin^2\lambda}\Big)\varepsilon^2 + O(\sigma^2)$$

that ε has the order of $\sigma^{m/2}$ as $\sigma \to 0$ on the level lines of the dispersion function $\Delta(0; F, \sigma, \lambda(F_2)) = C$ if the constant $C = M\sigma^m$ $(M > 0$ and $m > 0)$ has a power dependence on the small parameter σ. Therefore, in constructing the long-wave asymptotics, it should be taken into account that the original dimensional variable x has to be scaled $x/h_2 = \sigma^{-m/2}\, x_{m/2}$ ($x_{m/2}$ is the corresponding slow dimensionless variable) near the spectrum boundary on the level lines $\Delta = M\sigma^m$ with a given exponent m. In this sense, the dimensionless variable $x = x_{1/2}$ is used in the original equations (3)–(5) and in the basic approximate equation (8).

4. Bifurcation of Solitary Waves. The scale of the wavelength of the order $\sigma^{-1/2}$ taken into account in deriving Eq. (8) is natural for the points F on the level lines $\gamma_0(F, \sigma, \lambda(F_2)) = M\sigma$. For such points, extension of the unknown function $\eta = \sigma\eta_0$ in Eq. (12) in the lowest-order approximation yields the equation

$$\Big(\frac{d\eta_0}{dx_{1/2}}\Big)^2 = \eta_0^2(M + \gamma_1\eta_0). \tag{13}$$

Correspondingly, in the general case $(\gamma_1 \neq 0)$, the branching solitary waves have the form of the classical Korteweg–de Vries solitons with the amplitude of the order $O(\sigma)$, i.e., of the second order with respect to the parameter ε, which is the modulus of the wavenumber. Waves of elevation and depression are formed in the ranges of parameters satisfying the inequalities $\gamma_1 < 0$ and $\gamma_1 > 0$, respectively. Indeed, for $\gamma_1 \neq 0$, the order of smallness of the constant defining the level lines of the coefficient γ_0 is not very important. With the wave length scale consistent with the exponent $m > 0$, the limit equation retains the form of Eq. (13), and the wave amplitude is quadratic with respect to the wavenumber, independent of the choice of the family of level lines.

At points where the coefficient γ_1 vanishes, the contribution to the model equation can be made by terms with higher powers of η from the right side of Eq. (12). In addition, the form of the equation for the leading-order term can change, depending on the exponent of the constant defining the level lines of the coefficient γ_0. Let us find the approximate expressions for the coordinates of the bifurcation points of the special type mentioned, which are points of intersection of the curves $\gamma_0(F, \sigma, \lambda(F_2)) = 0$ and $\gamma_1(F, \sigma, \lambda(F_2)) = 0$ in the plane (F_1, F_2). Note, by virtue of the first equation of this system, the second equation is equivalent to the relation $\alpha_1(F, \sigma, \lambda(F_2)) = 0$. In turn, the approximate equation $\alpha_1(F, 0, \lambda(F_2)) = 0$ can be used instead of this relation. Using to the variable λ as an independent parameter and eliminating the Froude number F_1, we obtain an equation that has the following form with accuracy of the order $O(\sigma^2)$:

$$\frac{\sigma}{\mu}\Big(\frac{3\cot\lambda}{r\lambda} + 3\cot^2\lambda + 2\Big) - \frac{3}{r} = 0.$$

In the interval $\lambda \in (0, \pi)$, this equation has two roots asymptotically close to the ends of this interval: $\lambda_1 = \sqrt{(1+r)\sigma/\mu} + O(\sigma)$ and $\lambda_2 = \pi - \sqrt{r\sigma/\mu} + O(\sigma)$. In the limit, as $\sigma \to 0$, the root λ_1 corresponds to the point P_1 with the coordinates

$$F_1 = r/\sqrt{(1+r)r}, \qquad F_2 = 1/\sqrt{1+r}. \tag{14}$$

This point P_1 (Fig. 2a) is located in the first quadrant of the plane F; this is a bifurcation point for internal waves of the smooth bore type and plateau-shaped solitary waves in a two-layer fluid [7, 8]. Let us consider the level lines $\gamma_0(F, \sigma, \lambda(F_2)) = M\sigma^2$ in the vicinity of this point. Extension of the variables $x_1 = \sqrt{\sigma}x_{1/2}$ and $\eta = \sigma\eta_0$ in Eq. (8) leads to the following equation for the leading term of the asymptotics:

$$\Big(\frac{d\eta_0}{dx_1}\Big)^2 = \eta_0^2(M + \theta_1\eta_0 + \theta_2\eta_0^2). \tag{15}$$

Here $\theta_1 = \gamma'_{1\sigma}(F, 0, \lambda(F_2))$ and $\theta_2 = \gamma_2(F, 0, \lambda(F_2))$; γ_2 is the coefficient at η^2 in expansion (12). For $\theta_2 > 0$, Eq. (15) yields solutions of the solitary-wave type if the polynomial of the fourth power with respect to η_0 in its right side has two simple real non-zero roots $\eta_0 = a_1$ and $\eta_0 = a_2$ in addition to the double root $\eta_0 = 0$. If these roots coincide and yield one more double root $a = a_1 = a_2$ ($|a| = \sqrt{M/\theta_2}$), the wave on the interface has the form of a smooth bore with the profile $\eta_0(x_1) = (a/2)[1 + \tanh(\sqrt{M}\,x_1)]$. If the sample roots a_1 and a_2 are little different from each other, the solitary waves have the form of a broad plateau. In the limit, as $a_1 \to a_2$, the fronts of such a wave acquire the form of a smooth bore, which approximately describes one half of a symmetric plateau-shaped solitary wave. Thus, this particular case, as well as the above-considered general case, retains the basic asymptotic properties of weakly nonlinear waves typical for a standard two-layer model [10].

Let us now consider the root λ_2, which generates the point P_2 with the coordinates

$$F_1 = 1 + \frac{1}{2\pi}\sqrt{\frac{\sigma}{r\mu}} + O(\sigma), \qquad F_2 = \frac{1}{\pi}\sqrt{\frac{\sigma}{\mu}} + \frac{\sqrt{r}\,\sigma}{\pi^2\mu} + O(\sigma^{3/2}).$$

This point is located on the spectrum boundary in an immediate vicinity of the horizontal asymptote l of this boundary (Fig. 2b). The special properties of this bifurcation point are caused by the presence of weak continuous stratification in the upper layer of the fluid; this point has no analogs in the spectrum of the model with constant densities in the layers. In the vicinity of the branch of the curve Γ_1: $\gamma_1(F, 0, \lambda(F_2)) = 0$ passing through the point P_2, the solutions of Eq. (8) display an unusual behavior. We can demonstrate this fact by using approximate parametrization of the curve Γ_1

$$F_1 = 1 + [9r + \mu(3 - 2\pi^2 r^3)\tau^2 + 2\pi^2\mu^2 r^2\tau^4]\,\frac{\sqrt{\sigma}}{24\pi\mu r\tau}, \qquad F_2 = \frac{\sqrt{\sigma/\mu}}{\pi - \tau\sqrt{\sigma}}$$

with a parameter $\tau = \sqrt{r/\mu} + t\sqrt{\sigma}$. This parametrization is chosen so that the expressions for the Froude number at $t = 0$ give two first terms in the expansion of the coordinates of the point P_2 with respect to fractional powers of

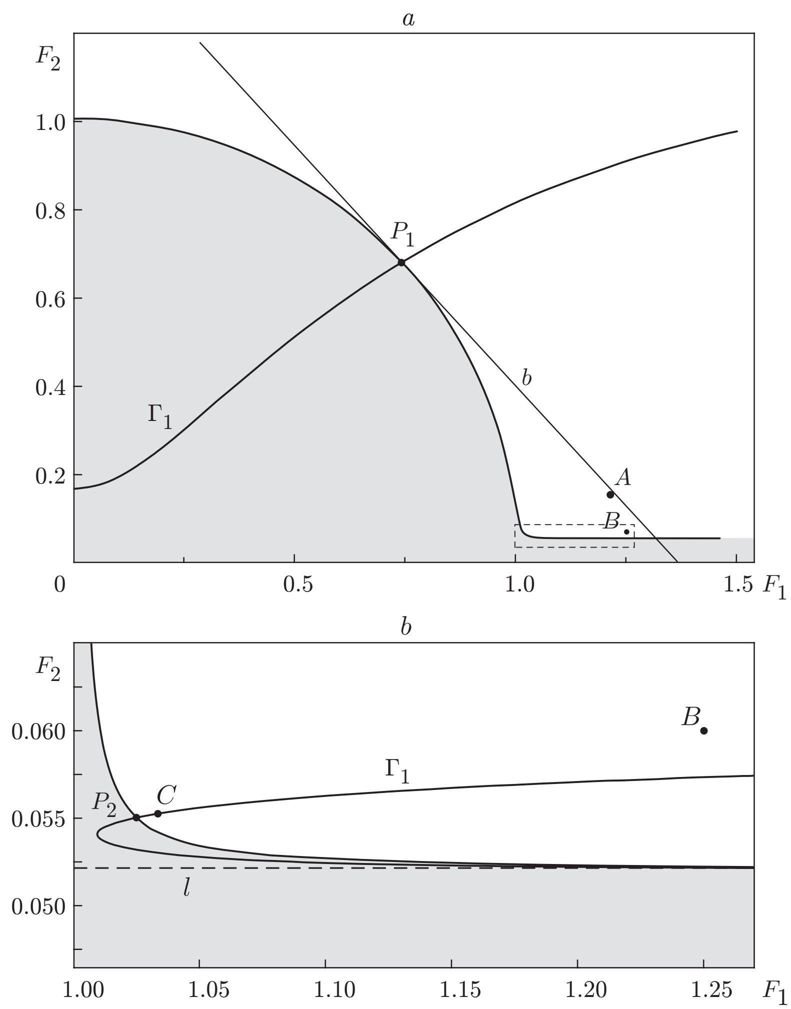

Fig. 2. Spectrum of the leading mode for $\sigma = 0.000\,08$, $\mu = 0.003$, and $r = 1.2$ (a) and a zoomed-in fragment of this spectrum (b).

the parameter σ. Applying the scaling of the variables $x = \sigma^{1/4} x_{1/4}$ and $\eta = \sigma^{1/2}\eta_0$ in Eq. (8) for such points F, we obtain the lowest-order equation

$$\Big(\frac{d\eta_0}{dx_{1/4}}\Big)^2 = \eta_0^3\,\frac{R^2 + 3R(\pi\eta_0 + R)^3 - 3(\pi\eta_0 + R)^4}{r^3(\pi\eta_0 + R)^4 + 3r^2/(2\pi^2\mu^2)}, \tag{16}$$

where $R = \sqrt{r/\mu}$. In contrast to Eqs. (13) and (15) considered above, the right side of Eq. (16) has a triple root $\eta_0 = 0$; for this reason, the solutions of Eq. (16) have a power rather than an exponential asymptotics of decay as $|x_{1/4}| \to \infty$. This is especially clearly seen for the simplified model equation $(d\eta_0/dx_{1/4})^2 = \alpha\eta_0^3(1 - \eta_0/\beta)$, where the constants α and β are fixed by linear interpolation of the rational function in the right side of Eq. (16) in the interval $(0, \eta_*)$, where η_* is the non-zero root of the numerator. The corresponding solution of the solitary-wave type has the form $\eta_0(x) = 4\beta/(4 + \alpha\beta x^2)$.

5. Wave Profiles. Let us consider the above-obtained asymptotics of small solutions in the context of approximate models of finite-amplitude waves, also related to Eq. (8). Far from the spectrum, the parameter $\varepsilon(\sigma)$ characterizing wave decay at infinity, generally speaking, is not small; therefore, in Eq. (8), we can pass to the dimensionless variable $x = x_0$ corresponding to identical vertical and horizontal linear scales. As Eq. (6) predicts that the parameter λ for fixed Froude numbers F_1 and $F_2 \neq 0$ has the order of smallness $\sqrt{\sigma}$, we expand the coefficients A_j and B_j with respect to the powers of λ and retain terms of the lowest power in the numerator and

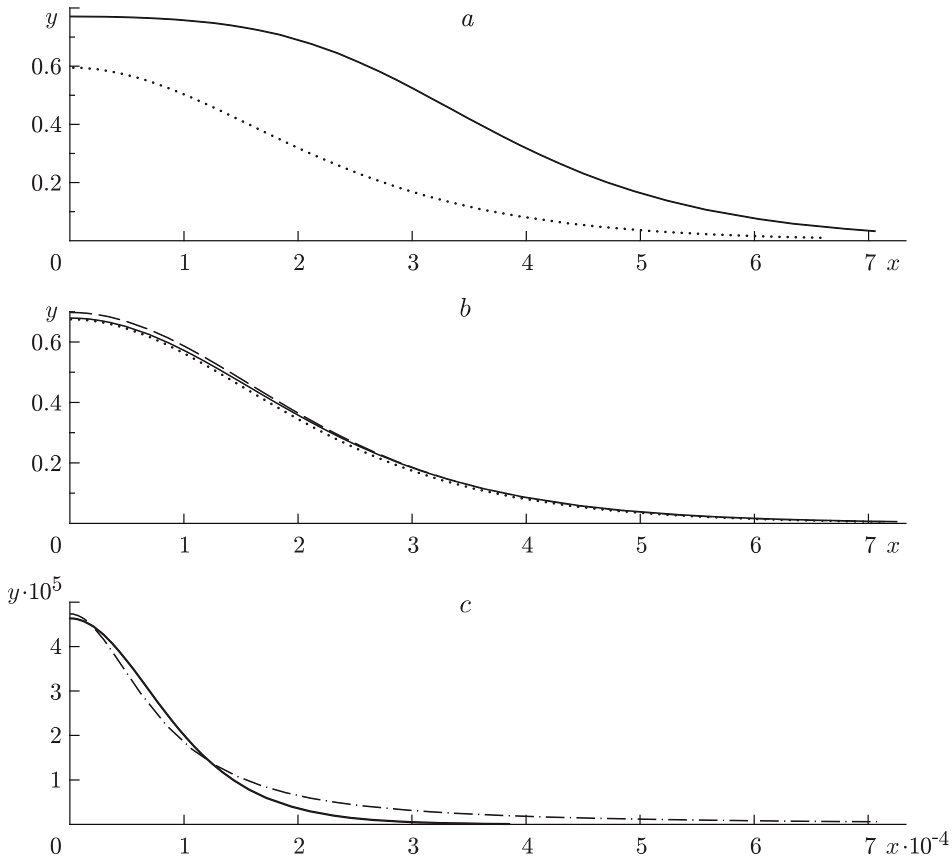

Fig. 3. Profiles of solitary waves for different points in the plane of the Froude numbers ($\sigma = 0.000\,08$, $\mu = 0.003$, and $r = 1.2$): (a) point A ($F_1 = 1.223$ and $F_2 = 0.15$); (b) point B ($F_1 = 1.25$ and $F_2 = 0.06$); (c) point C ($F_1 = 1.020\,84$ and $F_2 = 0.055\,03$); the solid curves show the solution of Eq. (8); the dotted curves show the solution of Eq. (18); the dashed curve is the solution of Eq. (17); the dash-and-dotted curve is the solution of Eq. (16).

denominator of the right side of Eq. (8). As a result of the limit transition as $\lambda \to 0$, we obtain an equation with a fractional-rational right side of the fourth power with respect to η

$$\Big(\frac{d\eta}{dx_0}\Big)^2 = \frac{3\eta^2[\eta^2 + (F_2^2 - rF_1^2 - 1 + r)\eta + r(F_1^2 + F_2^2 - 1)]}{r^3F_1^2(1-\eta) + F_2^2(r+\eta)}, \qquad (17)$$

which is the equation of the second-order approximation of the shallow water theory for a two-layer fluid with constant densities in the layers [3] written in dimensionless form. For points of the first quadrant of the plane (F_1, F_2), Eq. (17) has solutions of the solitary-wave type in the area of the supercritical domain $F_1^2 + F_2^2 > 1$ bounded by the straight line b: $\sqrt{r}F_1 + F_2 = \sqrt{1+r}$ (see Fig. 2a). The straight line b is the geometric location of points for which Eq. (17) has solutions of the smooth bore type. The bore bifurcates from the main flow at the tangential point P_1 of the straight line b and the spectrum boundary (unit circle). In beak-type domains in the vicinity of the bifurcation point with coordinates (14), the parameters of solitary waves are characterized by a special self-similar dependence on the Froude number [8]. In the model with weak continuous stratification, this property is manifested as stratification of a narrow region of weakly nonlinear asymptotics by level lines of the dispersion function with a power order $m = 2$ with respect to the parameter σ. Both for Eq. (8) and for Eq. (17), the equation for the leading term of the solution asymptotics near the point P_1 has the structure of Eq. (15). The two-layer fluid approximation (17) retains its accuracy with respect to model (8) in the domain including the points A and B (see Fig. 2a). The profiles of the solitary waves calculated for Eqs. (17) and (8) almost coincide (Fig. 3a).

In the area of the supercritical domain, which is located near the asymptote of the spectrum boundary (see Fig. 2b), the parameter λ is not small; according to Eq. (11), however, the Froude number F_2, which has the order of $O(\sqrt{\sigma}\,)$, turns out to be small. Taking this fact into account and making σ tend to zero in the right side of Eq. (8), we obtain

$$r^3F_1^2\eta_x^2 = 3\eta^2(r(F_1^2 - 1) - \eta). \qquad (18)$$

In its form, Eq. (18) coincides with the known Boussinesq–Rayleigh equation for solitary surface waves in a homogeneous fluid layer with a dimensionless depth r and Froude number $F_1 > 1$. This means that the leading mode of internal waves for Eq. (8) behaves here as the model of surface waves for a homogeneous lower fluid with the Froude number $F_1 = u_1/\sqrt{g_1h_1}$ [$g_1 = (\rho_1 - \rho_2)g/\rho_1$ is the reduced acceleration due to gravity]. This limit regime is also consistent with Eq. (17), which also yields Eq. (18) for $F_2 \to 0$. Figure 3b shows that the solitary wave profiles calculated by models (8), (17), and (18) for the point B (see Fig. 2), where the approximation of the two-layer fluid is still suitable, are in good agreement. Figure 3a shows that the area of applicability of the Boussinesq–Rayleigh equation is rather narrow as the Froude number F_2 increases. It is also of interest that this model rapidly loses accuracy as the Froude number F_1 decreases, because of approaching the singular point P_2 on the spectrum boundary. Figure 3c shows that the profiles of solitary waves for the basic model (8) and Eq. (16) are fairly close, whereas the wave amplitude for Eq. (18) differs by two orders of magnitude. For surface solitary waves, the Boussinesq–Rayleigh equation is known to provide the best fit in the vicinity of the bifurcation point $F_1 = 1$. In the case considered, this is prevented by the change in the leading order of nonlinearity in Eq. (8) on the curve Γ_1 near the point P_2. The neighborhood of this singular point has to be studied specially, because the effect of the absence of exponential asymptotics of decay of solitary waves for the approximate equation (16) on asymptotics of solutions of more generic models is not clear.

Conclusions. The effect of weak continuous stratification in one layer of a two-layer fluid on parameters of stationary waves on the interface is considered within the framework of the second-order approximation of the shallow water theory. It is demonstrated that branching of solitary waves of the leading mode from the basic piecewise-constant flow may follow one of the three scenarios, where the bifurcations are similar to the regime of branching of the classical Korteweg–de Vries solitary waves in the first case, and to the regime of solitary waves of the plateau-shaped and smooth bore types in a fluid with constant densities in the layers. The third type of branching observed for the Froude numbers $F_1 \approx 1$ and $F_2 = O(\sqrt{\sigma}\,)$ occurs only in the presence of continuous stratification. A typical feature for this scenario is the transition from the exponential decay of the solution as $|x| \to \infty$ to the power decay in the leading order with respect to σ near the bifurcation point. The transition to the parametric domain of finite-amplitude waves is consecutively described by a series of asymptotic models including the Boussinesq–Rayleigh model for small values of F_2 and the Ovsyannikov model for moderate values of F_2.

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REFERENCES

1. J. Turner, *Buoyancy Effects in Fluids*, University Press, Cambridge (1979).
2. C. S. Yih, *Stratified Flows*, Academic Press, New York (1980).
3. L. V. Ovsyannikov, N. I. Makarenko, V. I. Nalimov, et al., *Nonlinear Problems of the Theory of Surface and Internal Waves* [in Russian], Nauka, Novosibirsk (1985).
4. M. Miyata, "An internal solitary wave of large amplitude," *La Mer*, **23**, No. 2, 43–48 (1985).
5. A. G. Voronovich, "Strong solitary internal waves in a 2.5-layer model," *J. Fluid Mech.*, **474**, 85–94 (2003).
6. R. R. Long, "On the Boussinesq approximation and its role in the theory of internal waves," *Tellus*, **17**, No. 1, 46–52 (1965).
7. N. I. Makarenko, "Smooth bore in a two-layer fluid," *Int. Ser. Numer. Math.*, **106**, 195–204 (1992).
8. Zh. L. Mal'tseva, "On a class of internal solitary waves in a two-layer fluid," *J. Appl. Mech. Tech. Phys.*, **40**, No. 5, 824–830 (1999).
9. D. J. Benney and D. R. S. Ko, "The propagation of long large amplitude internal waves," *Stud. Appl. Math.*, **59**, 187–199 (1978).
10. T. Kakutani and N. Yamasaki, "Solitary waves on a two-layer fluid," *J. Phys. Soc. Jpn.*, **45**, 674–679 (1978).